**MATH1520 University Mathematics for Applications** **Spring 2021**

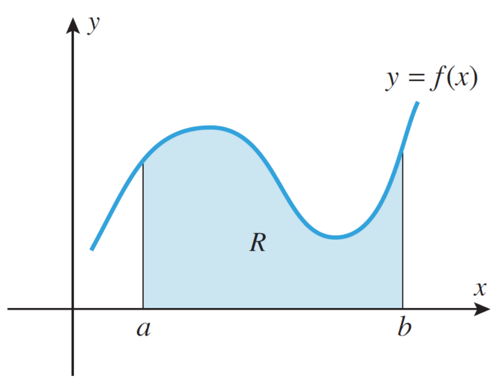
# Chapter 10: Definite Integrals

**Learning Objectives**:
(1) Define the definite integral and explore its properties.
(2) State the fundamental theorem of calculus, and use it to compute definite integrals.
(3) Use integration by parts and by substitution to find integrals.
(4) Evaluate improper integrals with infinite limits of integration.

## 1 Riemann Sums & Definite Integrals

Suppose $f$ is a function on $[a, b]$. Suppose further that $f(x)$ is positive on $[a, b]$. The we define

$$\int_a^b f(x)\, dx = \text{ area between } f(x) \text{ and the } x\text{-axis.}$$




What if some of the value of $f(x)$ is negative? Because $f(x)$ is negative, the "height" of $f(x)$ at this point is negative, so we take the area as negative. Therefore, we have the following definition.

**Definition 1.1** (Total Signed Area). Let $y = f(x)$ be defined on a closed interval $[a, b]$. The **total signed area from** $x = a$ **to** $x = b$ **under** $f$ is: the area under the graph of $f$ and above the $x$-axis on $[a, b]$ $-$ the area above the graph of $f$ and under the $x$-axis on $[a, b]$.

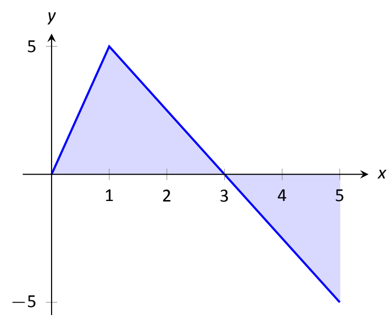
**Geometric interpretation of integration** The **definite integral of $f$ on** $[a, b]$ is the total signed area under $f$ on from $a$ to $b$, denoted

$$\int_a^b f(x)\,dx,$$

where $a$ and $b$ are the **bounds (or limits) of integration**.

We usually drop the word "signed" when talking about the definite integral, and simply say the definite integral gives "the area under the graph of $f$".

**Example 1.1.** Consider the function $f$ given below. Compute $\int_0^5 f(x)\,dx$.




*Solution.* The graph of $f$ is above the $x$-axis over $[0, 3]$. The area is $\frac{1}{2} \times 3 \times 5 = 7.5$.

The graph of $f$ is under the $x$-axis over $[3, 5]$. This is the "negative" area. The area is $-\frac{1}{2} \times 2 \times 5 = -5$. Hence

$$\int_0^5 f(x)dx = 7.5 - 5 = 2.5.$$

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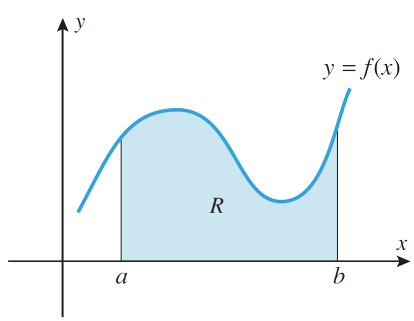
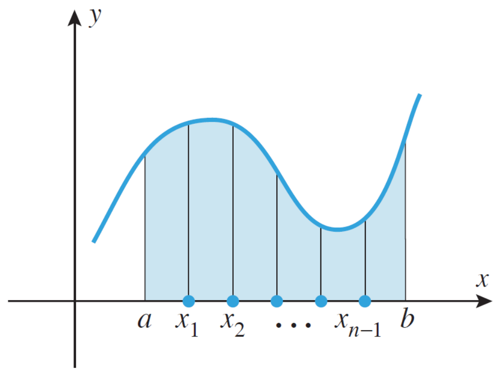
What if the region is not as simple as the previous example, such as the one below?



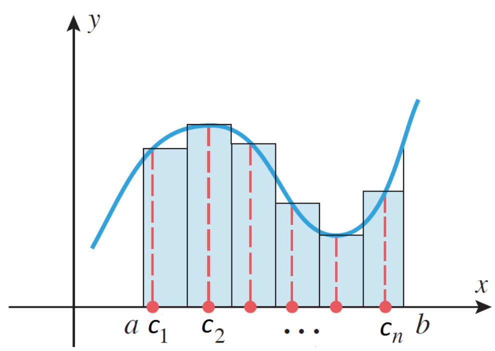
**Idea:** Approximate the area by small rectangles!

1. A partition of $[a, b]$: $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, $x_k = \dfrac{b-a}{n}k + a$, $k = 0, 1, \ldots, n$ divides $[a, b]$ into $n$ subintervals $[a_{k-1}, a_k]$ with width:

$$\Delta x_k = x_k - x_{k-1} = \frac{b-a}{n}, \quad k = 1, 2, \ldots, n.$$



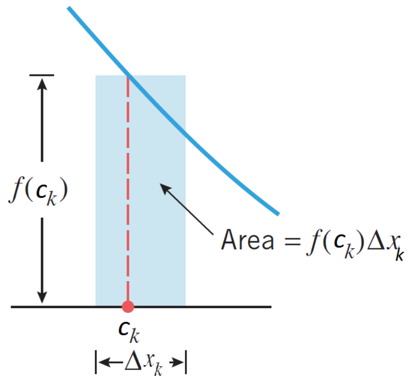

2. Choose points $c_k \in [x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$, to form small rectangles.




3. Calculate the area of each rectangle and sum them up.

   For the $k$th subinterval,

$$\boxed{\text{Area of } k\text{th rectangle} = \text{height} \times \text{width} = f(c_k)\Delta x_k}$$




**Definition 1.2.**

$$\boxed{\sum_{k=1}^{n} f(c_k)\Delta x_k}$$

is called a Riemann Sum of $f$ on $[a, b]$.

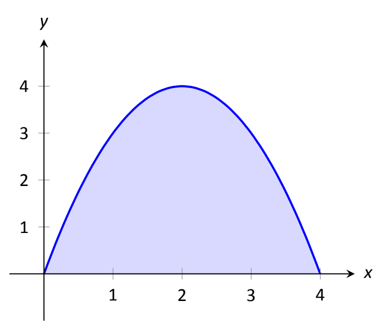
In particular,

if $c_k = x_{k-1}$, the sum is called left Riemann sum

if $c_k = x_k$, right Riemann sum

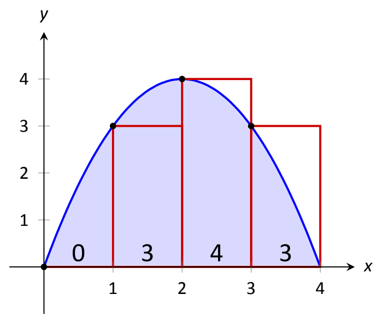
if $c_k = \dfrac{x_{k-1} + x_k}{2}$, mid-point Riemann sum

**Example 1.2.** Approximate the area under $y = 4 - x^2 + 4x$ on $[0, 4]$ with partition $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$.

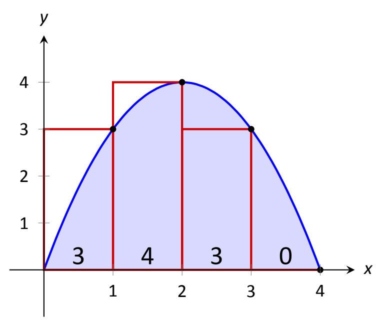



1. Left Riemann sum: $c_1 = 0, c_2 = 1, c_3 = 2, c_4 = 3$.

$$\text{Area} \approx f(0) \cdot 1 + f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 = 10.$$

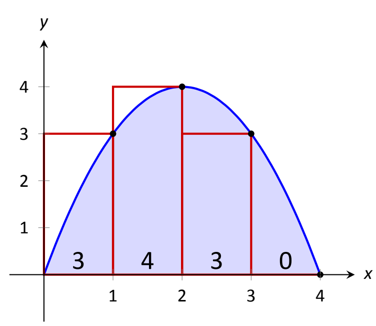



2. Right Riemann sum: $c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4.$




$$\text{Area} \approx f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 = 10.$$

3. Mid-point Riemann sum: $c_1 = , c_2 = 2, c_3 = 3, c_4 = 4.$




$$\text{Area} \approx f(0.5) \cdot 1 + f(1.5) \cdot 1 + f(2.5) \cdot 1 + f(3.5) \cdot 1 = 11.$$

Question: How to get better approximation of the area?

Solution: Increase number of rectangles.

**Definition 1.3.** Let $f(x)$ be continuous on $[a, b]$. Consider the partition: $x_k = \dfrac{b-a}{n}k + a$, $k = 0, 1, \ldots, n$. For any $c_k \in [x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$, $\lim\limits_{n\to+\infty} \sum\limits_{k=1}^{n} f(c_k)\Delta x_k$ is a fixed number, called definite (Riemann) integral of $f(x)$ on $[a, b]$, denoted by $\int_a^b f(x)\,dx$, i.e.,

$$\boxed{\lim_{n\to+\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)\,dx}$$

**Hard Theorem:** Let $f$ be a piecewise continuous function, then $\int_a^b f(x)\,dx$ is well-defined. I.e. The limit in the preceding definition exists, and is independent of the choices of $c_k$.

*Remark.* The "Lebesque integral" is well-defined for more general functions.

**Example 1.3.** Evaluate $\int_2^3 x\,dx$ using the left Riemann sum with $n$ equally spaced subintervals.

**Example 1.4.** Evaluate $\int_0^1 x^2\,dx$ using the right Riemann sum with $n$ equally spaced subintervals.

*Solution.* Consider the partition of $[0,1]$: $x_k = \frac{k}{n}, k = 0,\dots,n$.
Right Riemann sum: on $[x_{k-1}, x_k]$, $c_k = x_k = \dfrac{k}{n}$.

$$\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n}\left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{(n+1)(2n+1)}{6n^2}.$$

$(\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6})$

So, $\displaystyle\int_0^1 x^2\,dx = \lim_{n\to+\infty}\frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}.$ ■

*Remark.* It's so complicated to used definition to compute $\int_a^b f(x)\,dx$. Later, we will discuss another easier method: fundamental theorem of calculus.

**Theorem 1.1** (Properties of definite integrals)**.**

1. $\boxed{\int_a^a f(x)\,dx = 0}$
2. $\boxed{\int_a^b k\,dx = k(b-a)}$
3. $\boxed{\int_a^b (f(x)\pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx}$
   $\boxed{kf(x)\,dx = k\int_a^b f(x)\,dx}$
4. *if* $a < b$,
   $\boxed{\int_b^a f(x)\,dx \triangleq -\int_a^b f(x)\,dx}$ ($\triangleq$, *defined to be* )
5. $\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx}$
6. *if* $f(x) \le g(x)$ *on* $[a,b]$, *then*
   $\boxed{\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx}$

## 2 The fundamental Theorem of Calculus

Notation:

$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$: definite integral of function $f$ on $[a,b]$, which is a number.

$\int_a^x f(t)\,dt$: definite integral of function $f$ on $[a,x]$, it can be regarded as a function of $x$.

**Theorem 2.1** (Fundamental Theorem of Calculus)**.**

*Assume $f(x)$ is continuous.*

1. $\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$ *(i.e. $\int_a^x f(t)\,dt$ is an anti-derivative of $f(x)$)*
2. *Let $F(x)$ be any anti-derivative of $f(x)$, $F'(x) = f(x)$, then*
   $\boxed{\int_a^b f(x)\,dx = F(x)|_a^b := F(b) - F(a)}$.

*Remark.*

1. Differentiation $F'(x) = f(x)$ $\xleftrightarrow{\text{Fundamental thm of calculus}}$ Integration $\int_a^b f(x)\,dx = F(b) - F(a)$
2. Anti-derivative $F(x)$ is not unique. Which one should we choose?
   Another anti-derivative: $\tilde{F}(x) = F(x) + C$, then
   $$\tilde{F}(b) - \tilde{F}(a) = (F(b) + C) - (F(a) + C) = F(b) - F(a).$$
   so, it does not matter, we can choose any anti-derivative.

**Example 2.1.**

$$\int_1^9 \sqrt{x}\,dx = \int x^{1/2}\,dx\Big|_1^9 = \frac{2}{3}x^{3/2}\Big|_1^9 = \frac{2}{3}(27-1) = \frac{52}{3}.$$

**Example 2.2.** Evaluate $\int_1^2 \ln x\,dx$.

We first find one antiderivative of $\ln x$,

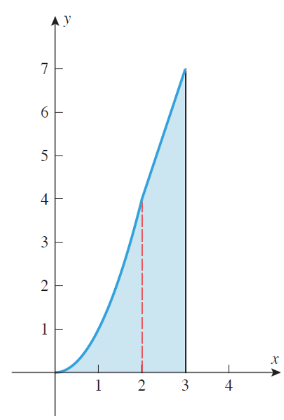
$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int 1 \, dx \qquad \text{(integration by parts)} \\ &= x \ln x - x + C. \end{aligned}$$

So, $\int_1^2 \ln x \, dx = (x \ln x - x)|_1^2 = 2 \ln 2 - 1$.

**Example 2.3.** Let

$$f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}.$$

Find $\int_0^3 f(x) \, dx$.



$$\begin{aligned}\int_0^3 f(x)\,dx &= \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx \qquad \text{(integrate separately)}\\ &= \frac{x^3}{3}\bigg|_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3 = \left(\frac{8}{3} - 0\right) + \left(\frac{15}{2} - 2\right) = \frac{49}{6}.\end{aligned}$$

**Exercise 2.1.**

1. $\displaystyle\int_0^1 2xe^{x^2}\,dx = e - 1.$
2. $\displaystyle\int_{-1}^2 |x|\,dx = \frac{5}{2}.$

**Example 2.4.** Compute $\dfrac{d}{dx}$ for (1) $\displaystyle\int_1^x e^{t^2}\,dt,$ (2) $\displaystyle\int_{x^2}^{x^3} e^{t^2}\,dt,$ (3) $\displaystyle\int_{g(x)}^{h(x)} f(t)\,dt.$

*Solution.* It's impossible to get explicit formula for $F(t) = \int e^{t^2}\,dt.$

1. By fundamental theorem of calculus (1), we have
$$\frac{d}{dx}\int_1^x e^{t^2}\,dt = e^{x^2}.$$
2. Let $F'(t) = e^{t^2}$, then
$$\frac{d}{dx}\int_{x^2}^{x^3} e^{t^2}\,dt = \frac{d}{dx}(F(x^3) - F(x^2)) = F'(x^3)\cdot 3x^2 - F'(x^2)\cdot 2x = e^{x^6}\cdot 3x^2 - e^{x^4}\cdot 2x.$$
3. Let $F'(t) = f(t)$,
$$\begin{aligned}\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt &= \frac{d}{dx}(F(h(x)) - F(g(x)))\\ &= F'(h(x))\cdot h'(x) - F'(g(x))\cdot g'(x)\\ &= f(h(x))h'(x) - f(g(x))g'(x).\end{aligned}$$

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**Exercise 2.2.** $\displaystyle\frac{d}{dx}\int_{2x}^{x+1} e^{\sqrt{t}}\,dt = e^{\sqrt{x+1}} - 2e^{\sqrt{2x}}.$

## 3 Definite Integration by Substitution & Integration by Parts

**Theorem 3.1.**

$$\int_a^b f(g(x))g'(x)\,dx \overset{g(x)=u}{=\!=\!=} \int_{g(a)}^{g(b)} f(u)\,du$$

**Example 3.1.**

1.

$$\begin{aligned}\int_0^1 8x(x^2+1)dx &= \int_0^1 4(x^2+1)\,d(x^2+1)\\ &= \int_1^2 4u\,du \quad (x^2+1=u, (0)^2+1=1, 1^2+1=2)\\ &= 2u^2\big|_1^2\\ &= 2\times 2^2 - 2\times 1^2 = 6.\end{aligned}$$

2.

$$\begin{aligned}\int_e^{e^2} \frac{1}{x\ln x}dx &= \int_e^{e^2} \frac{1}{\ln x}\,d(\ln x)\\ &= \int_1^2 \frac{1}{u}du \quad (\ln x = u, \ln e = 1, \ln e^2 = 2)\\ &= \ln u|_1^2\\ &= \ln 2 - \ln 1 = \ln 2.\end{aligned}$$

**Theorem 3.2.**

$$\int_a^b u(x)\,d(v(x)) = u(x)v(x)\Big|_a^b - \int_a^b v(x)\,d(u(x))$$

**Example 3.2.**

1.

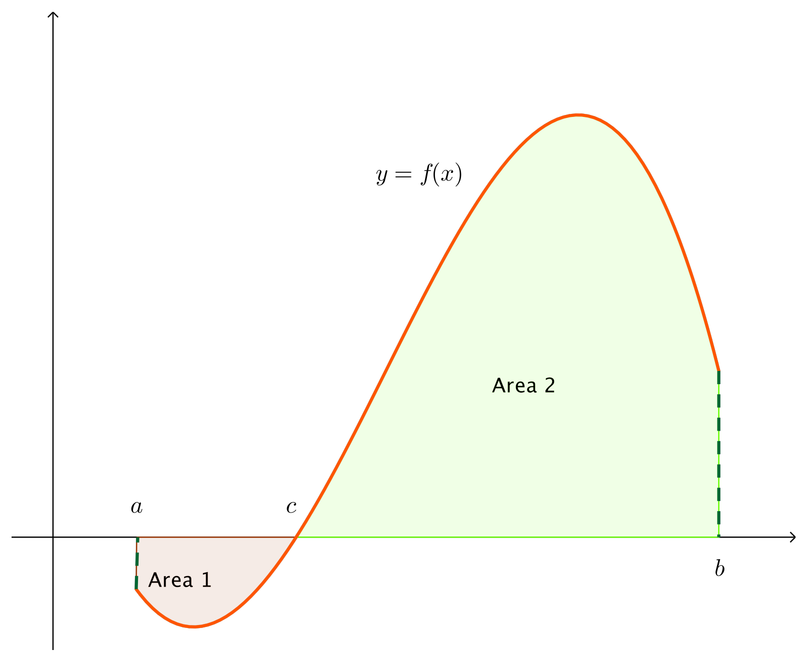
$$\begin{aligned}
\int_1^e x \ln x \, dx &= \int_1^e \ln x d\left(\frac{x^2}{2}\right) \\
&= \left[\frac{x^2}{2} \ln x\right]_1^e - \int_1^e \frac{x^2}{2} \, d \ln x \\
&= \left(\frac{e^2}{2} \ln e - \frac{1}{2} \ln 1\right) - \int_1^e \frac{x}{2} \, dx \\
&= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e \\
&= \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right) \\
&= \frac{e^2}{4} + \frac{1}{4}.
\end{aligned}$$

2.

$$\begin{aligned}
\int_0^1 xe^x \, dx &= \int_0^1 x \, d(e^x) \\
&= xe^x|_0^1 - \int_0^1 e^x \, dx \\
&= e - e^x|_0^1 = 1
\end{aligned}$$

# 4 Application of Definite Integration

## 4.1 Area bounded by $f(x)$ and $x$-axis on $[a, b] = \int_a^b |f(x)|\, dx$



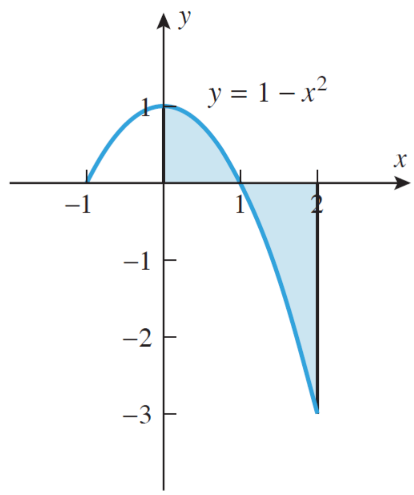

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx = -\text{ Area 1} + \text{ Area 2 } = \text{Signed area}$$

$$\int_a^b |f(x)|\, dx = \int_a^c -f(x)\, dx + \int_c^b f(x)\, dx = \text{ Area 1} + \text{ Area 2 } = \text{Area}$$

**Example 4.1.** Find the total area between the curve $y = 1 - x^2$ and the $x$-axis over the interval $[0, 2]$.

*Solution.* Let $1 - x^2 = 0, \quad \Rightarrow \quad x = \pm 1.$

$$1 - x^2 \begin{cases} \geq 0, & \text{for } -1 \leq x \leq 1, \\ < 0, & \text{for } x < -1 \quad \text{or} \quad x > 1. \end{cases}$$




The area is given by

$$\begin{aligned}\int_0^2 |1-x^2|\,dx &= \int_0^1 (1-x^2)dx + \int_1^2 -(1-x^2)dx \\ &= \left[x - \frac{x^3}{3}\right]_0^1 - \left[x - \frac{x^3}{3}\right]_1^2 \\ &= \frac{2}{3} - \left(-\frac{4}{3}\right) = 2.\end{aligned}$$

■

**Exercise 4.1.** Area bounded by $f(x) = x - \sqrt{x}$ and $x$-axis on $[0, 2]$.

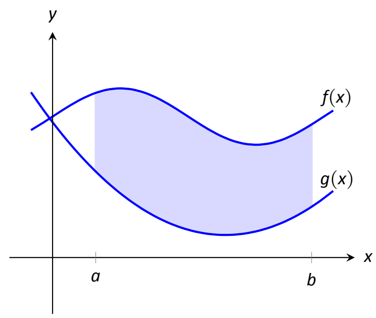
## 4.2 Area bounded by $f(x)$, $g(x)$ on $[a, b] = \int_a^b |f(x) - g(x)|\,dx$

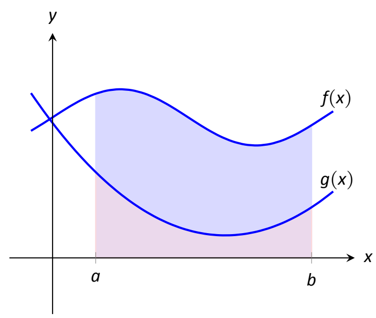
**Theorem 4.1.** *Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$ where $f(x) \geq g(x)$ for all $x$ in $[a, b]$. The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is*

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

*Proof.* The area between $f(x)$ and $g(x)$ is obtained by subtracting the area under $g$ from the area under $f$. Thus the area is

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x) - g(x))dx.$$

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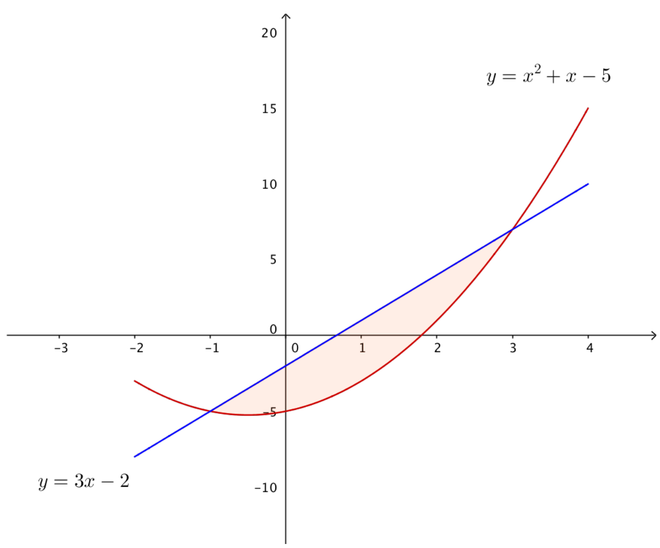
**Example 4.2.** Find the area of the region enclosed by $y = x^2 + x - 5$ and $y = 3x - 2$.




*Solution.* Let $x^2 + x - 5 = 3x - 2 \quad \Rightarrow \quad x = -1, 3.$

The area is

$$\begin{aligned}\int_{-1}^{3}\big((3x-2)-(x^2+x-5)\big)\,dx &= \int_{-1}^{3}(-x^2+2x+3)\,dx\\ &= \left(-\frac{1}{3}x^3+x^2+3x\right)\Bigg|_{-1}^{3}\\ &= -\frac{1}{3}(27)+9+9-\left(\frac{1}{3}+1-3\right)\\ &= 10\frac{2}{3}.\end{aligned}$$

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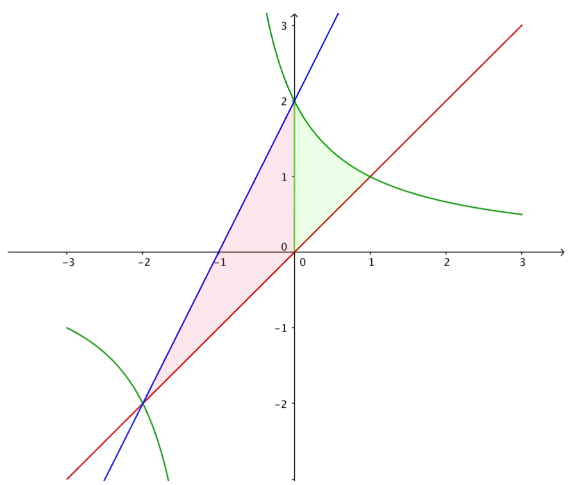
**Example 4.3.** Find the area bounded by

$$y=f(x)=x,\quad y=g(x)=\frac{2}{x+1},\quad \text{and}\quad y=h(x)=2x+2.$$




*Solution.* Area is

$$\begin{aligned}&\int_{-2}^{0}(h(x)-f(x))dx+\int_{0}^{1}(g(x)-f(x))dx\\ &=\int_{-2}^{0}(2x+2-x)+\int_{0}^{1}\left(\frac{2}{x+1}-x\right)dx\\ &=\left[\frac{x^2}{2}+2x\right]_{-2}^{0}+\left[2\ln|x+1|-\frac{x^2}{2}\right]_{0}^{1}\\ &=2+(2\ln 2-\frac{1}{2})=\frac{3}{2}+\ln 4.\end{aligned}$$

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## 4.3 Other Applications

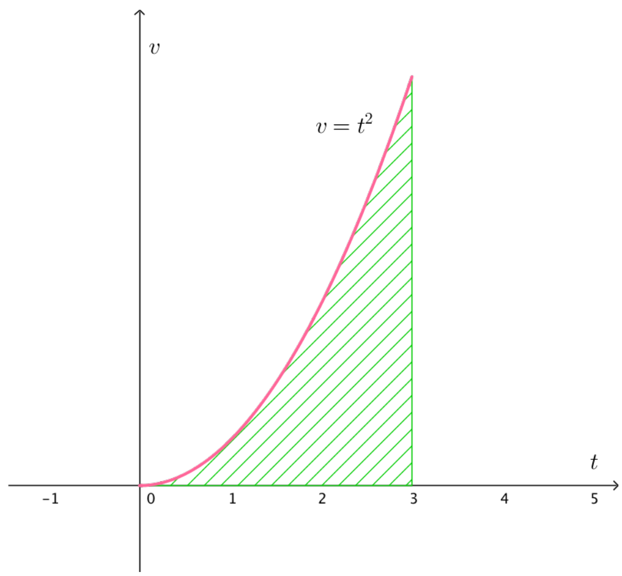
**Example 4.4.** An object moves along $x$-axis towards right with speed $v(t) = t^2$ m/s. Calculate the distance traveled from $t = 0$ to $t = 3$s.

*Solution.* Let $S(t)$ be the position at $t$. Then, $S'(t) = v(t) = t^2$.

The distance from $t = 0$ to $t = 3$ is

$$\underbrace{S(3) - S(0)}_{\text{total distance change}} = \int_0^3 \overbrace{S'(t)}^{\text{rate of change}} dt = \int_0^3 t^2 \, dt = \frac{1}{3}t^3 \Big|_0^3 = 9\text{m}$$

Geometrically,

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**Example 4.5.** Let $L(t)$ be the level of carbon monoxide (CO). Given that $L'(t) = 0.1t + 0.1$ parts per million (ppm). How much will the pollution change from $t = 0$ to $t = 3$?

*Solution.*

$$L(3) - L(0) = \int_0^3 L'(t)dt = \left[0.05t^2 + 0.1t\right]_0^1 = 0.75\text{ppm}.$$

■

**Exercise 4.2.** Let $t$ be the time (in hour). Let $m(t)$ be the mass of a certain amount of protein. The protein is changed to an amino acid and cause a decrease in mass at a rate

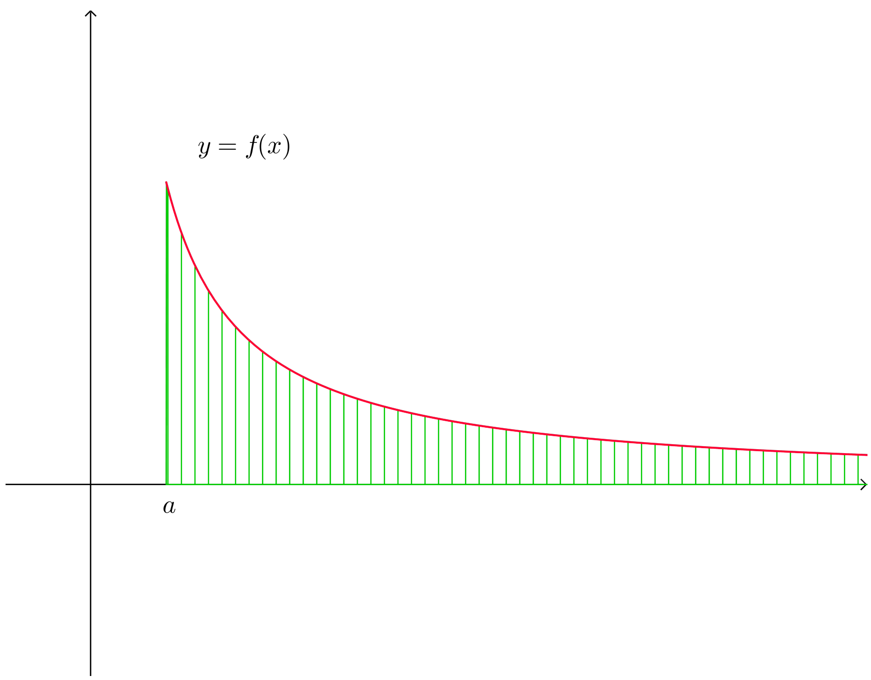
$$\frac{dm}{dt} = \frac{-2}{t+1}\text{g/hr.}$$

Find the decrease in mass of the protein from $t = 2$ to $t = 5$.

Ans: $-2\ln 2$.

# 5 Improper Integral

Question: How to find area of an unbounded region?




**Definition 5.1.** Define

1. $$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)dx$$

   if the limit exists, we say that the integral is **convergent**. Otherwise, **divergent**.

2. $$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$$

if the limit exists, we say that the integral is **convergent**. Otherwise, **divergent**.

3. Let $c$ be a fixed real number.

$$\boxed{\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx}$$

if both the two integrals on the right are convergent, we say that the integral is **convergent**. Otherwise, **divergent**.

**Example 5.1.**

1. $\displaystyle\int_0^{+\infty} e^{-x}dx = \lim_{b\to+\infty}\int_0^b e^{-x}dx = \lim_{b\to+\infty} -e^{-x}\Big|_0^b = \lim_{b\to+\infty}(e^0 - e^{-b}) = \lim_{b\to+\infty}(1-e^{-b}) = 1$, convergent.

2. $\displaystyle\int_1^{+\infty} \frac{1}{x}dx = \lim_{b\to+\infty}\int_1^b \frac{1}{x}dx = \lim_{b\to+\infty} \ln x|_1^b = \lim_{b\to+\infty} \ln b - \ln 1 = \lim_{b\to+\infty} \ln b = +\infty$, divergent.

3. $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\, dx = \lim_{b\to+\infty}\int_1^b \frac{1}{x^2}\, dx = \lim_{b\to+\infty}\left(-\frac{1}{x}\right)\Big|_1^b = \lim_{b\to+\infty}\left(1-\frac{1}{b}\right) = 1$, convergent.

4. $\displaystyle\int_1^{+\infty} \frac{1}{\sqrt{x}}\, dx = \lim_{b\to+\infty}\int_1^b \frac{1}{\sqrt{x}}\, dx = \lim_{b\to+\infty} 2\sqrt{x}\Big|_1^b = \lim_{b\to+\infty} 2(\sqrt{b}-1) = +\infty$, divergent.

**Example 5.2.** Compute $\displaystyle\int_0^{+\infty} \frac{dx}{(x+1)(3x+2)}$.

*Solution.*

$$\frac{1}{(x+1)(3x+2)} = \frac{3}{3x+2} - \frac{1}{x+1}.$$

Hence

$$\int_0^b \frac{dx}{(x+1)(3x+2)} = [\ln|3x+2| - \ln|x+1|]_0^b$$

$$= \ln|3b+2| - \ln|b+1| - \ln|2| = \ln\frac{|3b+2|}{|b+1|} - \ln 2.$$

Because

$$\lim_{b\to+\infty}\frac{|3b+2|}{|b+1|} = \lim_{b\to+\infty}\frac{|3b+2|\times\frac{1}{|b|}}{|b+1|\times\frac{1}{|b|}}.$$

$$\lim_{b\to+\infty} \frac{\left|3+\frac{2}{b}\right|}{\left|1+\frac{1}{b}\right|} = \frac{3}{1} = 3.$$

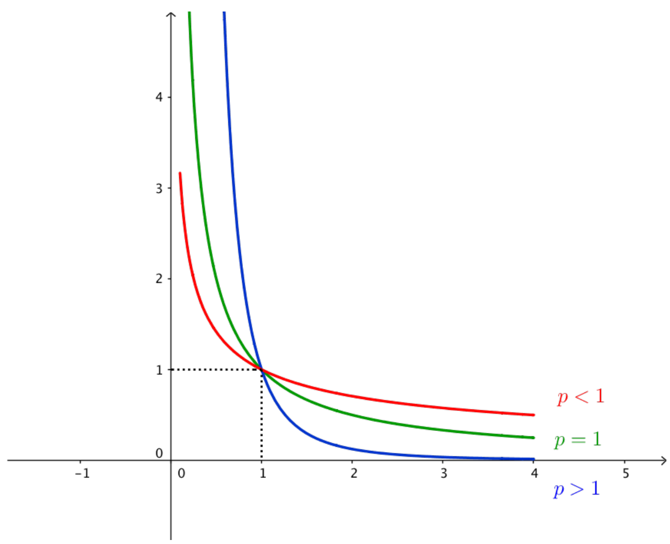
Therefore

$$\lim_{b\to+\infty} \int_0^b \frac{dx}{(x+1)(3x+2)} = \ln 3 - \ln 2.$$

■

**Exercise 5.1.** Let $p > 1$. Prove that

$$\int_1^{+\infty} \frac{1}{x^p}\, dx = \begin{cases} \dfrac{1}{p-1}, & \text{if} \quad p > 1, \qquad \text{convergent} \\ +\infty, & \text{if} \quad 0 < p \le 1, \qquad \text{divergent}. \end{cases}$$




*Remark.* From the above exercise,

1. $\lim_{x\to+\infty} f(x) = 0 \quad \not\Rightarrow \quad \int_1^{+\infty} f(x)\, dx$ is convergent.
2. For all $p > 0$, $\frac{1}{x^p} \to 0$ as $x \to +\infty$. However, only for $p > 1$, $\frac{1}{x^p}$ decays fast enough to guarantee the total area $\int_1^{+\infty} \frac{1}{x^p}\, dx$ is finite.

*Remark.* All the integration techniques can be applied, e.g. integration by substitution,...

**Example 5.3.** Compute $\int_{-\infty}^{1} xe^x\,dx$. (integration by parts)

*Solution.*

$$\int_{-\infty}^{1} xe^x\,dx = \lim_{a\to-\infty}\int_a^1 xe^x\,dx.$$

$$\int xe^x\,dx = \int x d(e^x) = xe^x - \int e^x\,dx = (x-1)e^x + C.$$

$$\begin{aligned}\int_{-\infty}^{1} xe^x\,dx &= \lim_{a\to-\infty}(x-1)e^x|_a^1\\ &= \lim_{a\to-\infty}(1-a)e^a \quad \infty\cdot 0 \quad \text{indeterminate form}\\ &= \lim_{a\to-\infty}\frac{1-a}{e^{-a}} \quad \frac{\infty}{\infty}\\ &= \lim_{a\to-\infty}\frac{-1}{-e^{-a}} \quad \text{L'Hôpital's rule}\\ &= 0.\end{aligned}$$

■

**Exercise 5.2.** $\int_{-\infty}^{1} x^2e^x dx = e$

**Example 5.4.** Compute $\int_{-\infty}^{+\infty} \frac{x}{(1+x^2)^2}\,dx$. (integration by substitution)

*Solution.* Using the substitution $u = 1+x^2$, we have

$$\int \frac{x}{(1+x^2)^2}\,dx = \frac{-1}{2(1+x^2)} + C.$$

Thus

$$\int_0^{+\infty} \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2}$$

and

$$\int_{-\infty}^{0} \frac{x}{(1+x^2)^2}\,dx = -\frac{1}{2}.$$

Hence

$$\int_{-\infty}^{+\infty} \frac{x}{(1+x^2)^2}\,dx = \int_0^{+\infty} \frac{x}{(1+x^2)^2}\,dx + \int_{-\infty}^{0} \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2} + \left(-\frac{1}{2}\right) = 0.$$

■